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COMMENT

# Comment on 'The relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{r}+f(r)L+g(r)=0$'

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**Abstract.** We comment on a recent paper by Leach and Gorringe. Our aim is to correct an incorrect statement and to examine why the groups $SO_4$ and $SU_3$ are the only compact dynamical groups for a particle in a spherical potential. The analysis is based on the distinction between Lie groups and Lie algebras and on a theorem due to Bertrand about closed orbits.

Recently, Leach and Gorringe (1990) made a study of the relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{r}+f(r)L+g(r)=0$. The results of their article cannot be understood if the word *symmetry* is not defined properly, that is with the aid of groups instead of algebras. Moreover, the role of the groups $SO_4$ and $SU_3$ can be understood once we relate them with a theorem due to the XIXth century mathematician Bertrand.

The statements presented by the authors in their introduction are misleading and, strictly speaking, even wrong. As an example, the first sentence states that all central potential problems possess the dynamical symmetries $SO_4$ and $SU_3$. The reader would be tempted to ask why $SO_4$ is usually associated with the $1/r$ potential and $SU_3$ with the three-dimensional isotropic oscillator rather than the converse. It is possible to clarify the situation with the aid of a few general properties.

Given a Hamiltonian $H$ on a six-dimensional phase space, if one is able to find $n$ observables which are Poisson commuting with $H$ and are a basis of a given Lie algebra, this will be also true for any other Hamiltonian; the reason is the following one: all symplectic manifolds of a given dimension are locally isomorphic (Bacry *et al* 1966). In this sense, the Lie algebra of $SO_4$ (or $SU_3$) is a 'symmetry algebra' for all problems with three degrees of freedom, even if the Hamiltonian is not associated with a central potential.

A kinematical symmetry is described by a *group* of spacetime transformations leaving the Hamiltonian invariant. The Noether theorem provides us, in such a case, with a set of conserved momenta. For instance, any spherical potential is responsible of the existence of the symmetry *group* $SO_3$ and, consequently, of the conserved angular momentum Lie *algebra*. According to the above statement on symplectic manifolds, any three-dimensional problem has three observables which are Poisson commuting with the Hamiltonian and forming a basis for the Lie algebra of $SO_3$. It does not follow however that any three-dimensional problem possesses the spherical symmetry $SO_3$. It is important to distinguish between group and algebraic symmetry or, in other words, between global and local symmetry.

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There exists a slightly generalized situation where a generalized kinematical symmetry can be defined. Instead of the Hamiltonian to be invariant, one can consider an invariant field of forces. Everybody knows, for instance, that a charged particle in the field of a magnetic monopole is a problem with spherical symmetry. The generalized Noether theorem provides us with a generalized angular momentum. It is clear that in this problem we have *also* $SO_3$ as a symmetry *group*.

A *dynamical* symmetry can be defined as a group $G_E$ of canonical transformations acting on the set of motions of given energy $E$. For two different energies $E$ and $E'$, the groups can be either isomorphic or distinct. As an example, for the isotropic three-dimensional oscillator, we have the symmetry group $SU_3$ for all values of the energy but for the Kepler problem, we have $SO_4$ for $E<0$, $SO_{3,1}$ for $E>0$, and the Euclidean group for $E=0$. The relationship between these groups is described by the contraction procedure (Inonü and Wigner 1953).

As we shall see, it is natural to mention the Bertrand theorem (Bertrand 1873) when we are concerned with spherical potentials. Let us state it. If the spherical potential $V(r)$ and its derivatives are continuous up to the third order, all bounded *trajectories are closed if the potential is of the form* $\frac{1}{2}m\omega^2r^2$ *or* $-k^2/r$.

I think that the reader will appreciate the following theorem (Bacry 1973). Suppose $\mathrm{d}V(r)/\mathrm{d}r>0$ and consider the set of motions of energy $H$. The length of the angular momentum $|\boldsymbol{L}|$ is extremum for circular trajectories. This extremum is a maximum provided $r^3(\mathrm{d}V(r)/\mathrm{d}r)$ is an increasing function. In that case, $\boldsymbol{L}^2$ takes all values from zero (rectilinear motions) to $\boldsymbol{L}^2_{\max}=mr^3(\mathrm{d}V(r)/\mathrm{d}r)$ (circular motions).

This theorem permits to clarify the relationship between $SO_4$ and the two potentials $-k^2/r$ and $\frac{1}{2}m\omega^2r^2$. For both, the condition of the theorem is satisfied. Because a circular trajectory satisfies $H=\frac{1}{2}r(\mathrm{d}V(r)/\mathrm{d}r)+V(r)$, one obtains

$$\boldsymbol{L}^2_{\max}=-\frac{mk^4}{2H} \tag{1}$$

for the Kepler potential and

$$\boldsymbol{L}^2_{\max}=\frac{H^2}{\omega^2}. \tag{2}$$

Let us now define the vector $\boldsymbol{A}$ as follows. It lies in the direction of the aphelion and has its length given by

$$\boldsymbol{A}^2=\boldsymbol{L}^2_{\max}-\boldsymbol{L}^2.$$

Any Kepler motion of a given negative energy $H$ is *uniquely* described by the two orthogonal vectors $\boldsymbol{L}+\boldsymbol{A}$ and $\boldsymbol{L}-\boldsymbol{A}$. These two vectors define a set of two points on a sphere of radius $\boldsymbol{L}^2_{\max}=\boldsymbol{L}^2+\boldsymbol{A}^2$. If follows that the set of motions we are interested in is isomorphic to the topological product $S^2\times S^2$. This is a homogeneous space of $SO_4$, namely $SO_4/(U_1\times U_1)$. That is why $SO_4$ is a dynamical group for the bounded motions of the Kepler problem.

The situation is different for the harmonic oscillator *because we now have two aphelions. That is why we cannot distinguish between the two vectors* $\boldsymbol{L}+\boldsymbol{A}$ *and* $\boldsymbol{L}-\boldsymbol{A}$. The set of motions of given energy $H$ is the symmetrized product of two spheres $(S^2\times S^2)/Z^2$, which is a homogeneous space of $SU_3$, namely $SU_3/U_2$.

The role of the Bertrand theorem is obvious. If there was another spherical potential for which all bounded trajectories were closed, the continuity conditions would imply that all these trajectories would have a fixed number of aphelions, say $n$. The set of bounded motions of energy $H$ would be the symmetrized product of $n$ spheres $S^2$, which is a homogeneous space of the group $SU_{n+1}$, namely the homogeneous space $SU_{n+1}/(U_1\times U_1\times\ldots\times U_1)$ (see Bacry 1980). Bertrand's theorem says that the only spherical potential which has $SO_4$ as a symmetry group is the Kepler potential.

A final remark concerns the quantization problem. As shown in Souriau (1970), the quantization of the sphere is obtained in making its radius equal to an integer. From (1), one gets the quantization of the hydrogen atom, namely $H=-mk^2/2n^2$, as expected. The same procedure applied to (2) would give $H^2=n^2\omega^2$, that is $H=n\omega$, instead of $H=(n+\frac{3}{2})\omega$. There must be an explanation for this discrepancy, but I do not know it. It would be interesting to understand a formal use of such a procedure in the general case.

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